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ON THE HEIGHT OF THE KRONECKER PRODUCT OF S_n CHARACTERS

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ABSTRACT

Given any two heights h_1, h_2, we can choose wide enough partitions $\nu, \mu \in \mathrm{Par}(n)$ such that $h(\nu) = h_1$, $h(\mu) = h_2$ and $h(\chi_\nu \otimes \chi_\mu) = h_1 \cdot h_2$.

Introduction

The partitions of $n, \lambda \in \mathrm{Par}(n)$, are in a one-to-one correspondence with the irreducible characters of $S_n : \lambda \leftrightarrow \chi_\lambda$. The height $h(\lambda)$ of the corresponding Young diagram D_λ is defined to be the height of $\chi_\lambda : h(\chi_\lambda) = h(\lambda)$. For $\psi_n = \sum_{\lambda \in \mathrm{Par}(n)} m_\lambda \chi_\lambda$, any S_n character, let

$$h(\psi_n) = \max\{h(\lambda) \mid \lambda \in \mathrm{Par}(n), m_\lambda \neq 0\}.$$

In [4] and [6] we raised

QUESTION H. Given two heights h_1, h_2, is there an $N = N(h_1, h_2)$ such that for any $n \geqq N$ there are two S_n characters χ_ν, χ_μ $(\nu, \mu \in \mathrm{Par}(n))$ satisfying $h(\chi_\nu) = h_1$, $h(\chi_\mu) = h_2$ and $h(\chi_\nu \otimes \chi_\mu) = h_1 \cdot h_2$?!

Applying P.I. theory we then answered "H" affirmatively when $h_1 = k^2$, $h_2 = l^2$ are squares, and conjectured "yes" to "H" in general.

In this note the conjecture is proved for any two heights (Theorem 4). The proof, which uses asymptotic methods and no P.I. theory, also yields a polynomial rate of growth for the multiplicities m_λ of some irreducible characters χ_λ in $\chi_\nu \otimes \chi_\mu$, with $h(\lambda) = h_1 \cdot h_2$. The asymptotic results for degrees of Young diagrams are developed in [3], [5].

The field F is of characteristic zero. Since S_n has the same character theory over all such fields, assume F is algebraically closed (for example, choose $F = \mathbf{C}$).

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The introduction closes with a short summary of some basic facts about the representation theory of S_n and of $\mathrm{GL}(k)$ (see [1]). Let V be a vector space of dimension $k = \dim_F V$, then S_n and $\mathrm{GL}(V) = \mathrm{GL}(k)$ act on $W = V^{\otimes n}$ in natural ways: $\sigma \in S_n$, $u_1 \otimes \cdots \otimes u_n \xrightarrow{\sigma} u_{\sigma(1)} \otimes \cdots \otimes u_{\sigma(n)}$, $T \in \mathrm{GL}(V)$, $u_1 \otimes \cdots \otimes u_n \xrightarrow{T} Tu_1 \otimes \cdots \otimes Tu_n$. The images of S_n and of $\mathrm{GL}(V)$ in $\mathrm{End}(V^{\otimes n}) = E$ span two corresponding subalgebras:

$$\begin{array}{ccc} S_n & & \mathrm{GL}(V) \\ \downarrow & & \downarrow \\ A(k,n), & & B(k,n) \subseteq E. \end{array}$$

By Schur's theory they centralize each other in $E : \mathrm{Hom}_{FS_n}(V^{\otimes n}, V^{\otimes n}) = B(k,n)$, and vice versa. Also,

$$\dim B(k,n) = \binom{n + k^2 - 1}{n}.$$

Finally, let $\nu \in \mathrm{Par}(n)$ satisfy $h(\nu) \leqq \dim V$. By a theorem of Weyl, $V^{\otimes n}$ contains an FS_n irreducible submodule $W_\nu \subseteq V^{\otimes n}$ with character $\chi(W_\nu) = \chi_\nu$. In fact, $V^{\otimes n}$ contains the direct sum of exactly ${}^{(k)}N_\nu$ such FS_n submodules, where ${}^{(k)}N_\nu$ is the degree of the irreducible character of $\mathrm{GL}(k)$ that corresponds to the partition ν.

The main results

Let $\Lambda_k(n) = \{\lambda \in \mathrm{Par}(n) \mid h(\lambda) \leqq k\}$. It is shown in [4] that if $\nu, \mu \in \mathrm{Par}(n)$, then

$$\chi_\nu \otimes \chi_\mu = \sum_{\lambda \in \Lambda_{h(\nu)\cdot h(\mu)}(n)} m_\lambda \chi_\lambda.$$

The following lemma has interest on its own.

LEMMA 1. *Let $\nu, \mu \in \mathrm{Par}(n)$, $h(\nu) = h_1$, $h(\mu) = h_2$, and write $\chi_\nu \otimes \chi_\mu = \sum_{\lambda \in \Lambda_{h_1 \cdot h_2}(n)} m_\lambda \chi_\lambda$, then for all λ,*

$$m_\lambda \leqq \binom{n + (h_1 \cdot h_2)^2 - 1}{n}^{1/2}.$$

PROOF. Choose spaces V_1, V_2 with $\dim_F V_i = h_i$, $i = 1, 2$. Let $W_i = V_i^{\otimes n}$, then

$$W = (V_1 \otimes V_2)^{\otimes n} \underset{\substack{\uparrow \\ [\text{as } FS_n \text{ modules}]}}{\simeq} (V_1^{\otimes n}) \otimes (V_2^{\otimes n}) = W_1 \otimes W_2.$$

Now, $V_1^{\otimes n} \supseteq W_\nu$, an FS_n submodule with $\chi(W_\nu) = \chi_\nu$. Similarly, $V_2^{\otimes n} \supseteq W_\mu$, $\chi(W_\mu) = \chi_\mu$, so that $\chi(W_\nu \otimes W_\mu) = \chi_\nu \otimes \chi_\mu = \Sigma_\lambda m_\lambda \chi_\lambda$. Thus $W_\nu \otimes W_\mu = \Sigma_\lambda \bigoplus K_\lambda$, where each K_λ is a direct sum of m_λ copies of FS_n irreducible modules, each with character χ_λ. By Schur's lemma [see 2, §2],

$$\mathrm{Hom}_{FS_n}(W_\nu \otimes W_\mu, W_\nu \otimes W_\mu) \cong \sum_\lambda F_{m_\lambda}$$

(where F_m are the $m \times m$ matrices over F). Thus

$$\dim_F(\mathrm{Hom}_{FS_n}(W_\nu \otimes W_\mu, W_\nu \otimes W_\mu)) = \sum_\lambda m_\lambda^2,$$

hence

$$\begin{aligned} m_\lambda &\leqq [\dim_F(\mathrm{Hom}_{FS_n}(W_\nu \otimes W_\mu, W_\nu \otimes W_\mu))]^{\frac{1}{2}} \\ &\leqq [\dim_F(\mathrm{Hom}_{FS_n}(W_1 \otimes W_2, W_1 \otimes W_2))]^{\frac{1}{2}} \\ &= [\dim_F(\mathrm{Hom}_{FS_n}(W, W))]^{\frac{1}{2}} = \binom{n + (h_1 \cdot h_2)^2 - 1}{n}^{\frac{1}{2}}, \end{aligned}$$

since $W = (V_1 \otimes V_2)^{\otimes n}$ and $\dim V_1 \otimes V_2 = h_1 \cdot h_2$. Q.E.D.

REMARK 2. W_1 contains a direct sum of ${}^{(h_1)}N_\nu$ FS_n irreducible submodules with character χ_ν, and similarly for W_2. Thus a similar proof yields

$$m_\lambda \leqq \frac{1}{{}^{(h_1)}N_\nu \cdot {}^{(h_2)}N_\mu} \cdot \binom{n + (h_1 \cdot h_2)^2 - 1}{n}^{\frac{1}{2}}.$$

In fact one gets

$$\sum_\lambda m_\lambda^2 \leqq \left(\frac{1}{{}^{(h_1)}N_\nu \cdot {}^{(h_2)}N_\mu}\right)^2 \binom{n + (h_1 \cdot h_2)^2 - 1}{n}.$$

REMARK 3. Let $\nu, \mu \in \mathrm{Par}(n)$, $h(\nu) \leqq h_1$, $h(\mu) \leqq h_2$, then fix h_1, h_2 and send $n \to \infty$. Lemma 1 implies that the multiplicities in $\chi_\nu \otimes \chi_\mu$ are bounded above by a polynomial in n of degree $\leqq \frac{1}{2}[(h_1 \cdot h_2)^2 - 1]$. In the next theorem we find a lower bound for some of these m_λ's, which is also of a polynomial rate of growth.

The essential tool for proving Theorem 4 is the asymptotic result ([5, 4.5])

$$S_l^{(1)}(n) \simeq b_l \left(\frac{1}{n}\right)^{l(l-1)/4} \cdot l^n,$$

where

$$b_l = l^{l(l-1)/4} \cdot \frac{1}{l!} \cdot \Gamma\left(\frac{3}{2}\right)^{-l} \cdot \prod_{j=1}^{l} \Gamma(1 + \tfrac{1}{2}j) \quad \text{(a constant)}$$

and $S_l^{(1)}(n) \stackrel{\text{def}}{=} \Sigma_{\lambda \in \Lambda_l(n)} d_\lambda$.

THEOREM 4. *Let h_1, h_2 be any two heights $\geqq 2$, then there exists $N = N(h_1, h_2)$ such that for any $n \geqq N$ there is a pair of partitions $\nu, \mu \in \mathrm{Par}(n)$ (in fact many pairs) with $h(\nu) = h_1$, $h(\mu) = h_2$ and $h(\chi_\nu \otimes \chi_\mu) = h_1 \cdot h_2$.*

Moreover, let e be any number satisfying $e \lneqq \frac{1}{4}h_1 \cdot h_2(h_1h_2 - 1) - \frac{1}{2}(h_1^2 + h_2^2) + 1$, and write $\chi_\nu \otimes \chi_\mu = \sum_{\lambda \in \Lambda_{h_1 \cdot h_2}(n)} m_\lambda \chi_\lambda$, then there are some $\lambda \in \Lambda_{h_1 \cdot h_2}(n)$ with $h(\lambda) = h_1 \cdot h_2$ and for which $m_\lambda \geqq n^e$.

PROOF. First, let $n = h_1\omega_1 = h_2\omega_2$ and consider the $h_1 \times \omega_1$ and $h_2 \times \omega_2$ rectangles $\nu = (\omega_1^{h_1})$ and $\mu = (\omega_2^{h_2})$. By [3, §3], as $n \to \infty$,

$$d_\nu \simeq a_{h_1} \cdot \left(\frac{1}{n}\right)^{(h_1^2-1)/2} \cdot h_1^n \quad \text{and} \quad d_\mu \simeq a_{h_2} \cdot \left(\frac{1}{n}\right)^{(h_2^2-1)/2} \cdot h_2^n,$$

where $a_h = (h-1) \cdot (h-2)^2 \cdots 2^{h-2} \cdot (1/\sqrt{2\pi})^{h-1} \cdot h^{(h^2+1)/2}$. Thus

$$\mathrm{degree}(\chi_\nu \otimes \chi_\mu) = d_\nu \cdot d_\mu \simeq a \cdot \left(\frac{1}{n}\right)^{(h_1^2+h_2^2)/2-1} \cdot (h_1 \cdot h_2)^n,$$

where $a = a_{h_1} \cdot a_{h_2}$.

Write

$$\chi_\nu \otimes \chi_\mu = \sum_{\lambda \in \Lambda_{h_1 \cdot h_2 - 1}(n)} m_\lambda \chi_\lambda + \sum_{h(\lambda) = h_1 \cdot h_2} m_\lambda \chi_\lambda,$$

let $e < \frac{1}{4}h_1 \cdot h_2(h_1 \cdot h_2 - 1) - \frac{1}{2}(h_1^2 + h_2^2) + 1$ and assume $m_\lambda \leqq n^e$ for all large n and all λ, $h(\lambda) = h_1 \cdot h_2$. By Lemma 1,

$$\text{all } m_\lambda \leqq p(n) = \binom{n + (h_1 \cdot h_2)^2 - 1}{n}^{\frac{1}{2}},$$

so that

$$\begin{aligned}
d_\nu \cdot d_\mu &= \deg(\chi_\nu \otimes \chi_\mu) \leqq p(n) \cdot S^{(1)}_{h_1h_2-1}(n) + n^e \sum_{h(\lambda)=h_1 \cdot h_2} d_\lambda \\
&\leqq p(n) \cdot S^{(1)}_{h_1h_2-1}(n) + n^e \cdot S^{(1)}_{h_1h_2}(n) \\
&\simeq b_{h_1h_2-1} \cdot \frac{p(n)}{q(n)} \cdot (h_1h_2 - 1)^n + b_{h_1h_2} \cdot n^e \cdot \left(\frac{1}{n}\right)^{h_1h_2(h_1h_2-1)/4} \cdot (h_1h_2)^n \\
&\quad (q(n) = n^{(h_1h_2-1)(h_1h_2-2)/4}) \\
&\simeq b_{h_1h_2} \cdot n^e \cdot \left(\frac{1}{n}\right)^{h_1h_2(h_1h_2-1)/4} \cdot (h_1h_2)^n.
\end{aligned}$$

Thus

$$a \cdot \left(\frac{1}{n}\right)^{(h_1^2+h_2^2)/2-1} \cdot (h_1 \cdot h_2)^n \lesssim b_{h_1h_2} \cdot \left(\frac{1}{n}\right)^{h_1h_2(h_1h_2-1)/4-e} \cdot (h_1h_2)^n$$

or

$$n^{(h_1h_2)(h_1h_2-1)/4-(h_1^2+h_2^2)/2+1} \lesssim c \cdot n^e \qquad (c = b_{h_1h_2}/a = \text{constant})$$

for all large n, a contradiction. This proves the theorem for "rectangular" partitions ν, μ (with $h_1 > 2$ or $h_2 > 2$).

For more general partitions, apply (F.1.1) of [5]: Let $\lambda \in \Lambda_h(n, a, \delta)$ (see there), then for large n,

$$d_\lambda \simeq \gamma_n \cdot D(c) \cdot e^{-hc^2/2} \cdot \left(\frac{1}{n}\right)^{(h-1)(h+2)} \cdot h^n.$$

The previous proof can now be applied to various other pairs of partitions $\lambda^{(i)} \in \Lambda_{h_i}(n, a, \delta)$, $i = 1, 2$. Notice that the exponent $\frac{1}{2}(h_1^2 + h_2^2) - 1$ is now replaced by nearly its half, namely by $\frac{1}{4}[(h_1-1)(h_1+2)+(h_2-1)(h_2+2)]$. If

$$\bar{e} < \tfrac{1}{4}(h_1h_2)(h_1h_2-1) - \tfrac{1}{4}[(h_1-1)(h_1+2)+(h_2-1)(h_2+2)]$$

then for some $\lambda \in \Lambda_{h_1h_2}(n) - \Lambda_{h_1h_2-1}(n)$, $m_\lambda > n^{\bar{e}}$ for all large n. Q.E.D.

Remark 5. It seems that with some more effort, $N(h_1, h_2)$ can explicitly be computed. Notice that when $h_1 = k^2$, $h_2 = l^2$, then by P.I. theory, $N(h_1, h_2) \leqq 2k^2l^2 - 1$ ([4], [6]).

From the proof of the general case of Theorem 4 it is clear that $N(h_1, h_2)$ depends on the rate of growth of $d_{\lambda^{(i)}}$ as $n \to \infty$. Thus, to make $N(h_1, h_2)$ smaller, one should look for $\lambda^{(i)} \in \Lambda_{h_i}(n, a, \delta)$ with maximal degrees $d_{\lambda^{(i)}}$. Such partitions will be studied elsewhere.

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